

# Summing parquet diagrams using the functional renormalization group: X-ray problem revisited

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We present a simple method for summing so-called parquet diagrams of fermionic many-body systems with competing instabilities using the functional renormalization group. Our method is based on partial bosonization of the interaction multi-channel Hubbard-Stratonovich transformations. A simple truncation of the resulting flow equations, retaining only the frequency-independent parts of the two-point and three-point vertices amounts to solving coupled Bethe-Salpeter equations for the effective interaction to leading logarithmic order. We apply our method by revisiting the X-ray problem and deriving the singular frequency dependence of the X-ray response function and the particle-particle susceptibility. Our method is quite general and should be useful in many-body problems involving strong fluctuations in several scattering channels.

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## I. INTRODUCTION

The basic structure of two-particle Green functions of interacting quantum many-body systems can exhibit singularities for different combinations of external momenta and frequencies. In simple cases only one particular combination dominates the perturbation series, so that it is sufficient to resum the diagrams corresponding to this combination. For example, slightly above a superconducting instability the two-particle Green function becomes singular if the total momentum and the total energy of the two incoming particles are small (particle-particle channel). It is then sufficient to approximate the effective interaction, appearing in the skeleton diagram of the two-particle Green function, by summing only the ladder diagrams with small total momentum<sup>1</sup>. Another example is the electron gas at high densities, where the effective interaction can be calculated by summing the geometric series of particle-hole bubbles which dominate for small momentum transfers<sup>1</sup>. In some cases, however, a single dominant scattering channel does not exist and singularities in more than one scattering channel appear in perturbation theory. In such cases, the usual strategy of summing only particle-particle ladders or particle-hole bubbles is inapplicable and one has to solve coupled Bethe-Salpeter equations in more than one channel. Diagrammatically, this means summing the so-called parquet diagrams where particle-hole bubbles and particle-particle ladders are self-consistently inserted into each other. Historically, such a resummation was first used by Sudakov and co-authors<sup>2,3</sup> in the context of high-energy meson-meson scattering. Since then, parquet methods have played an important role in studying quantum impurity models<sup>4-9</sup>, low-dimensional metals<sup>10-13</sup>, liquid Helium<sup>14-17</sup>, and vortex liquids<sup>18</sup>. Moreover parquet methods have also been used to construct approximations to reduced density matrices of large physical systems<sup>19</sup>, nuclear structure calculations<sup>20</sup>, and different lattice models for strongly correlated electrons<sup>21-28</sup>.

In particular, in two seminal papers by Roulet *et al.*<sup>5</sup> and Nozières *et al.*<sup>6</sup> the threshold exponents of the so-called X-ray problem<sup>29</sup> were obtained using the parquet method. The agreement of the parquet result for the threshold exponents with the known exact result<sup>30</sup> at weak coupling supports the validity of this technique.

While parquet methods are a well established many-body tool<sup>31,32</sup>, they are not straightforward to apply to physical problems of interest. One reason could be related to the fact that the derivation of coupled Bethe-Salpeter equations in several channels often relies on subtle diagrammatic and combinatoric considerations. Moreover, the explicit solution of coupled Bethe-Salpeter equations often requires rather substantial algebraic manipulations or extensive numerical calculations. It is well known, however, that the summation of parquet diagrams can also be formulated in terms of the renormalization group. In fact, the modern formulation of the Wilsonian renormalization group for fermionic many-body systems in terms of a formally exact hierarchy of flow equations for the one-particle irreducible vertices<sup>33-36</sup> (we shall refer to this approach as the functional renormalization group, abbreviated by FRG) opens new possibilities for dealing with parquet type of problems. On the one hand, the FRG offers a general and systematic framework of deriving parquet equations in a purely algebraic way without relying on diagrammatic arguments; on the other hand, the formulation of coupled Bethe-Salpeter equations as FRG flow equations offers new strategies of obtaining approximate solutions.

In this work we shall revisit the X-ray problem and present a systematic approach to obtain the well-known parquet results<sup>5,6</sup> within the framework of the FRG. In the X-ray problem both the particle-particle and the particle-hole channel exhibit logarithmic singularities, so that the calculation of the X-ray response function can be used as a benchmark for testing truncations of the FRG flow equations. Moreover the approximations made in Refs. [5 and 6] rely on the logarithmic nature of the

singularities. However, the necessary algebra is quite demanding. Here we show that the threshold exponents of the X-ray problem and the frequency dependence of the corresponding response function can be obtained in a relatively straightforward manner within the FRG. Our particular implementation of the FRG relies on the partial bosonization of the theory, in both singular scattering channels, using multi-channel Hubbard Stratonovich transformations. We believe that this technique will also be useful in other problems which are dominated by competing singularities in several channels.

The rest of the paper is organized as follows. In section II we briefly present the X-ray problem and formulate it using functional integrals. In section III we identify the cut-off scheme for the FRG and introduce the regularized deep hole Green function. We write down the exact FRG flow equation for the deep hole self-energy and calculate the leading term in the weak coupling expansion of the anomalous dimension of the  $d$ -electrons. In the succeeding section IV, we present our FRG-based approach to obtain the particle-hole susceptibility, describing the X-ray response, and also derive a similar expression for the corresponding particle-particle susceptibility. In the concluding section V we summarize our obtained results.

## II. FUNCTIONAL INTEGRAL FORMULATION OF THE X-RAY PROBLEM

Before we present the X-ray problem in the functional integral framework, we shall briefly describe the nature of the problem. The X-ray absorption and emission in metals is modeled by the following second quantized Hamiltonian<sup>29</sup>

$$\mathcal{H} = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} c_{\mathbf{k}}^{\dagger} c_{\mathbf{k}} + \epsilon_d d^{\dagger} d + \sum_{\mathbf{k}\mathbf{k}'} U_{\mathbf{k}\mathbf{k}'} c_{\mathbf{k}}^{\dagger} c_{\mathbf{k}'} d^{\dagger} d, \quad (1)$$

where  $c_{\mathbf{k}}^{\dagger}$  creates a conduction electron with momentum  $\mathbf{k}$  and energy  $\epsilon_{\mathbf{k}} = \mathbf{k}^2/(2m)$ , while  $d^{\dagger}$  creates a localized deep core electron with energy  $\epsilon_d < 0$ . The Coulomb interaction between the conduction electrons is neglected and only the intraband part of the Coulomb interaction between the conduction electrons and the deep core electron is taken into account. For simplicity, the spin degrees of freedom are ignored and a separable interaction of the form  $U_{\mathbf{k}\mathbf{k}'} = U u_{\mathbf{k}} u_{\mathbf{k}'}$  is assumed, where the form factors  $u_{\mathbf{k}}$  will be specified below.

We are interested in the  $d$ -electron Green function

$$G^d(t - t') = -i \langle \mathcal{T}[d(t) d^{\dagger}(t')] \rangle. \quad (2)$$

Moreover, the experimentally measurable X-ray transition rate is determined by the Fourier transform of the particle-hole response function

$$\chi^{\text{ph}}(t - t') = \langle \mathcal{T}[\hat{A}(t) \hat{A}(t')] \rangle, \quad (3)$$

where  $\mathcal{T}$  is the time ordering symbol and the composite particle-hole operator  $\hat{A}(t)$  is defined by

$$\hat{A}(t) = [c^{\dagger}(t) d(t) + d^{\dagger}(t) c(t)], \quad (4)$$

with

$$c(t) = \sum_{\mathbf{k}} u_{\mathbf{k}} c_{\mathbf{k}}(t). \quad (5)$$

The Fourier transforms  $G^d(\omega)$  and  $\chi^{\text{ph}}(\omega)$  of the above functions for real frequencies  $\omega$  in the vicinity of certain threshold frequencies  $\omega_d$  and  $\omega_x$  are known to be of the form<sup>5,6,29</sup>

$$G^d(\omega) \sim \frac{1}{\omega - \omega_d} \left( \frac{\omega - \omega_d}{\xi_0} \right)^{\eta}, \quad (6)$$

$$\chi^{\text{ph}}(\omega) \sim \frac{\nu_0}{2u} \left[ \left( \frac{\xi_0}{\omega - \omega_x} \right)^{\alpha} - 1 \right], \quad (7)$$

where  $\nu_0$  is the density of states of conduction electrons at the Fermi energy and  $\xi_0$  is an ultraviolet cutoff of the order of the width of the conduction band. For small values of the dimensionless interaction  $u = \nu_0 U$  the threshold exponents are given by

$$\eta = u^2 + \mathcal{O}(u^3), \quad (8)$$

$$\alpha = 2u + \mathcal{O}(u^2). \quad (9)$$

In Refs. [5 and 6] the above results were derived by explicitly writing down and solving parquet equations which resum the leading logarithmic singularities in both the particle-hole and the particle-particle channel. In the following we show that the FRG offers an alternative and, in our opinion, simpler way to derive these results.

In order to set up the FRG for the X-ray problem, it is useful to start from an effective Euclidean action depending only on the degrees of freedom corresponding to the  $d$ -electrons and the relevant linear combination (5) of the conduction electron operators. To derive this, consider the Euclidean action associated with the Hamiltonian (1),

$$\begin{aligned} S = & \int_0^{\beta} d\tau \sum_{\mathbf{k}} \bar{c}_{\mathbf{k}}(\tau) (\partial_{\tau} + \xi_{\mathbf{k}}) c_{\mathbf{k}}(\tau) \\ & + \int_0^{\beta} d\tau \bar{d}(\tau) (\partial_{\tau} + \xi_d) d(\tau) \\ & + U \int_0^{\beta} d\tau \bar{c}(\tau) c(\tau) \bar{d}(\tau) d(\tau), \end{aligned} \quad (10)$$

where the energies  $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu$  and  $\xi_d = \epsilon_d - \mu$  are measured relative to the chemical potential  $\mu$ . Although at this point we keep the inverse temperature  $\beta$  finite, later we will take the zero temperature limit  $\beta \rightarrow \infty$  whenever it is convenient. The quantities  $c_{\mathbf{k}}(\tau)$  and  $d(\tau)$  are now Grassmann variables depending on imaginary time  $\tau$ , and the Grassman variable  $c(\tau)$  represents the linear combination  $c(\tau) = \sum_{\mathbf{k}} u_{\mathbf{k}} c_{\mathbf{k}}(\tau)$ . To derive an effective action depending only on  $c(\tau)$ , we integrate over all  $c_{\mathbf{k}}$ -variables with the exception of the linear combination  $c(\tau)$  of interest. Thus we insert the following representation of unity into the functional integral representation

of the correlation functions,

$$\begin{aligned} 1 &= \int \mathcal{D}[\bar{c}', c'] \prod_{\tau} \delta(\bar{c}'(\tau) - \bar{c}(\tau)) \delta(c'(\tau) - c(\tau)) \\ &= \int \mathcal{D}[\bar{c}', c'] \int \mathcal{D}[\bar{\eta}, \eta] e^{\int_0^{\beta} d\tau [\bar{\eta}(c' - c) + (\bar{c}' - \bar{c})\eta]}, \quad (11) \end{aligned}$$

where  $c'$  and  $\eta$  are new Grassmann variables and the functional delta-function of the Grassmann variables is defined as usual<sup>37</sup>. Then we integrate over the  $c_{\mathbf{k}}$ -variables and subsequently over the auxiliary  $\eta$ -variables. After renaming  $c' \rightarrow c$  we Fourier transform the Gaussian part to frequency space and obtain the effective impurity model

$$\begin{aligned} S_{\text{imp}} &= - \int_{\omega} (G_0^c(i\omega))^{-1} \bar{c}_{\omega} c_{\omega} - \int_{\omega} (G_0^d(i\omega))^{-1} \bar{d}_{\omega} d_{\omega} \\ &+ U \int_0^{\beta} d\tau \bar{c}(\tau) c(\tau) \bar{d}(\tau) d(\tau), \quad (12) \end{aligned}$$

where the symbol  $\int_{\omega} = \beta^{-1} \sum_{\omega}$  denotes summation over fermionic Matsubara frequencies  $i\omega$  such that for vanishing temperature  $\int_{\omega} = \int \frac{d\omega}{2\pi}$ , and the Fourier components of the fields are defined by  $c_{\omega} = \int_0^{\beta} d\tau e^{i\omega\tau} c(\tau)$  and  $d_{\omega} = \int_0^{\beta} d\tau e^{i\omega\tau} d(\tau)$ . The Gaussian part of the action (12) depends on the non-interacting Green functions

$$G_0^c(i\omega) = \sum_{\mathbf{k}} \frac{u_{\mathbf{k}}^2}{i\omega - \xi_{\mathbf{k}}}, \quad (13)$$

$$G_0^d(i\omega) = \frac{1}{i\omega - \xi_d}. \quad (14)$$

At this point it is convenient to assume that the form factor  $u_{\mathbf{k}}$  is only finite for  $|\xi_{\mathbf{k}}| < \xi_0$ , where  $\xi_0$  is a bandwidth cutoff of the order of the Fermi energy  $\epsilon_F = \mu$ . Moreover, we also assume that  $u_{\mathbf{k}}$  is such that

$$\sum_{\mathbf{k}} \frac{u_{\mathbf{k}}^2}{i\omega - \xi_{\mathbf{k}}} = \nu_0 \int_{-\xi_0}^{\xi_0} d\xi \frac{1}{i\omega - \xi}, \quad (15)$$

where  $\nu_0$  is the density of states at the Fermi energy. The above integral becomes elementary and we obtain

$$G_0^c(i\omega) = -i\pi\nu_0 \text{sgn}\omega \left[ 1 - \frac{2}{\pi} \arctan\left(\frac{|\omega|}{\xi_0}\right) \right]. \quad (16)$$

The effective impurity model defined by Eq. (12) is the starting point of our FRG calculation.

### III. DEEP HOLE GREEN FUNCTION

We begin by discussing the FRG calculation of the  $d$ -electron (deep hole) Green function. It turns out that the leading order term in the weak coupling expansion of the corresponding anomalous dimension  $\eta$  can be obtained in a straightforward way without resumming parquet diagrams. To set up the FRG procedure, we need to specify

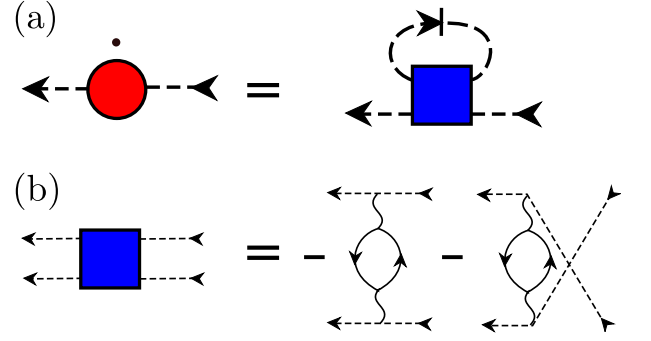


FIG. 1. (Color online) (a) Graphical representation of the exact FRG flow equation (19) for the  $d$ -electron self-energy. The dashed arrows denote the  $d$ -electron propagators, the dashed arrow with an extra slash denotes the  $d$ -electron single-scale propagator, the shaded (red) circle with a dot represents the scale derivative of the  $d$ -electron self-energy, and the shaded (blue) square represents the one-particle irreducible effective interaction between the  $d$ -electrons. (b) Feynman diagrams contributing to the effective interaction (25) between the  $d$ -electrons to second order in the bare interaction. The solid arrows forming the loops represent the non-interacting conduction electron propagators.

our cutoff scheme. Since the propagator of the  $c$ -fermions does not exhibit any singularity [see Eq. (16)], we do not introduce any cutoff in this sector. On the other hand, we regularize the singularity of the non-interacting  $d$ -electron propagator by replacing

$$(G_0^d(i\omega))^{-1} \rightarrow (G_{0\Lambda}^d(i\omega))^{-1} = i\omega - \xi_d + R_{\Lambda}^d(i\omega), \quad (17)$$

where the regulator function  $R_{\Lambda}^d(i\omega)$  vanishes for  $\Lambda \rightarrow 0$  and diverges for  $\Lambda \rightarrow \infty$ . The regularized  $d$ -electron propagator is then

$$G_{\Lambda}^d(i\omega) = \frac{1}{i\omega - \xi_d - \Sigma_{\Lambda}^d(i\omega) + R_{\Lambda}^d(i\omega)}, \quad (18)$$

where the cutoff-dependent  $d$ -electron self-energy  $\Sigma_{\Lambda}^d(i\omega)$  satisfies the exact FRG flow equation<sup>33-36</sup>

$$\partial_{\Lambda} \Sigma_{\Lambda}^d(i\omega) = \int_{\omega'} \dot{G}_{\Lambda}^d(i\omega') \Gamma_{\Lambda}^{\bar{d}d\bar{d}d}(\omega, \omega'; \omega', \omega). \quad (19)$$

Here

$$\dot{G}_{\Lambda}^d(i\omega) = [-\partial_{\Lambda} R_{\Lambda}^d(i\omega)][G_{\Lambda}^d(i\omega)]^2 \quad (20)$$

is the so-called single-scale propagator. The irreducible four-point vertex  $\Gamma_{\Lambda}^{\bar{d}d\bar{d}d}(\omega'_1, \omega'_2; \omega_2, \omega_1)$  can be identified with the properly antisymmetrized effective interaction between two  $d$ -electrons for a given value of the cutoff  $\Lambda$ . A graphical representation of the flow equation (19) is shown in Fig. 1 (a). A convenient regulator for the  $d$ -electrons is a Litim regulator<sup>38</sup> in frequency space,

$$R_{\Lambda}^d(i\omega) = i \text{sgn}\omega (\Lambda - |\omega|) \Theta(\Lambda - |\omega|), \quad (21)$$

so that the cutoff derivative of the regulator is

$$\partial_{\Lambda} R_{\Lambda}^d(i\omega) = i \text{sgn}\omega \Theta(\Lambda - |\omega|). \quad (22)$$

For simplicity we choose the zero of energy such that the  $d$ -electron energy vanishes. Ignoring self-energy corrections to the  $d$ -electron propagator, we get

$$G_{\Lambda}^d(i\omega) \approx \begin{cases} \frac{1}{i\Lambda \text{sgn}\omega} & \text{for } |\omega| < \Lambda, \\ \frac{1}{i\omega} & \text{for } |\omega| > \Lambda, \end{cases} \quad (23)$$

and

$$\dot{G}_{\Lambda}^d(i\omega) \approx \frac{i \text{sgn}\omega}{\Lambda^2} \Theta(\Lambda - |\omega|). \quad (24)$$

Since we do not impose any cutoff on the  $c$ -fermion propagator, the effective interaction  $\Gamma_{\Lambda}^{\bar{d}dd}(\omega'_1, \omega'_2; \omega_2, \omega_1)$  between the  $d$ -electrons is finite even at the initial scale  $\Lambda = \Lambda_0$ . It should be pointed out that our bare action (10) does not contain an interaction of this type. Diagrammatically, the effective interaction  $\Gamma_0^{\bar{d}dd}(\omega'_1, \omega'_2; \omega_2, \omega_1)$  at the initial scale  $\Lambda = \Lambda_0$  is determined by all diagrams with four external  $d$ -legs and only  $c$ -propagators in internal loops. In the weak coupling regime, we may approximate the flowing  $\Gamma_{\Lambda}^{\bar{d}dd}(\omega'_1, \omega'_2; \omega_2, \omega_1)$ , on the right-hand side of Eq. (19), by its initial value at  $\Lambda_0$ . For the calculation of the leading term in the weak coupling expansion of the anomalous dimension  $\eta$  of the  $d$ -electrons, we only need the effective interaction to second order in the bare interaction  $U$ . For the special frequencies  $\omega'_1 = \omega_1 = \omega$  and  $\omega'_2 = \omega_2 = \omega'$  appearing on the right-hand side of the FRG flow equation (19) we obtain the effective interaction between the  $d$ -electrons up to second order in the bare interaction

$$\Gamma_0^{\bar{d}dd}(\omega, \omega'; \omega', \omega) = -U^2 [\Pi_0^{cc}(i\omega - i\omega') - \Pi_0^{cc}(0)] + \mathcal{O}(U^3), \quad (25)$$

where  $\Pi_0^{cc}(i\bar{\omega})$  is the particle-hole bubble with non-interacting  $c$ -fermion propagators,

$$\Pi_0^{cc}(i\bar{\omega}) = \int_{\omega} G_0^c(i\omega - i\bar{\omega}) G_0^c(i\omega). \quad (26)$$

The Feynman diagrams contributing to Eq. (25) are shown in Fig. 1 (b). For small  $|\omega|$  we may approximate  $G_0^c(i\omega) \approx -i\pi\nu_0 \text{sgn}\omega$  and obtain

$$\Gamma_0^{\bar{d}dd}(\omega, \omega'; \omega', \omega) = -\pi(\nu_0 U)^2 |\omega - \omega'| + \mathcal{O}(U^3). \quad (27)$$

Substituting this effective interaction into the right-hand side of the exact FRG flow equation (19) we obtain a closed integro-differential equation for the frequency dependent  $d$ -electron self-energy  $\Sigma_{\Lambda}^d(i\omega)$ . From the numerical solution of this equation one can obtain the  $d$ -electron propagator  $G_{\Lambda}^d(i\omega)$  for all frequencies. Fortunately, the singular behavior close to the threshold can be extracted from the leading term in the low-energy expansion of the self-energy,

$$\Sigma_{\Lambda}^d(i\omega) = \Sigma_{\Lambda}^d(0) - (1 - Z_{\Lambda}^{-1})i\omega + \mathcal{O}(\omega^2), \quad (28)$$

where the cutoff-dependence of the wave-function renormalization factor  $Z_{\Lambda}$  defines the flowing anomalous dimension,

$$\eta_{\Lambda} = \Lambda \partial_{\Lambda} \ln Z_{\Lambda}. \quad (29)$$

This quantity can be obtained from the right-hand side of the exact FRG flow equation as follows<sup>35</sup>,

$$\begin{aligned} \eta_{\Lambda} &= Z_{\Lambda} \Lambda \lim_{\omega \rightarrow 0} \frac{\partial}{\partial(i\omega)} \partial_{\Lambda} \Sigma_{\Lambda}^d(i\omega) \\ &= Z_{\Lambda} \Lambda \int_{\omega'} \dot{G}_{\Lambda}^d(i\omega') \lim_{\omega \rightarrow 0} \frac{\partial}{\partial(i\omega)} \Gamma_{\Lambda}^{\bar{d}dd}(\omega, \omega'; \omega', \omega). \end{aligned} \quad (30)$$

Substituting our perturbative second order result (27) and our lowest order approximation (24) for the single-scale propagator in Eq. (30) we obtain

$$\eta = \lim_{\Lambda \rightarrow 0} \eta_{\Lambda} = (\nu_0 U)^2 + \mathcal{O}(U^3), \quad (31)$$

in agreement with the known weak coupling expansion (8). Finally, keeping in mind that our low-energy expansion (28) is only valid for  $|\omega| \lesssim \Lambda$ , we may estimate the  $d$ -electron propagator from

$$G^d(i\omega) \approx \frac{Z_{\Lambda=|\omega|}}{i\omega} \approx \frac{(|\omega|/\Lambda_0)^{\eta}}{i\omega}. \quad (32)$$

Recalling that we have set the threshold energy  $\omega_d$  equal to zero, we recover the known threshold behavior (6) of the  $d$ -electron propagator, where we identify  $\xi_0 = \Lambda_0$ .

#### IV. X-RAY AND PARTICLE-PARTICLE RESPONSE

In this section, using the FRG formalism, we demonstrate how to obtain the non-perturbative result (7) for the particle-hole susceptibility  $\chi^{\text{ph}}(\omega)$  which describes the X-ray response. Moreover, we shall also derive an analogous expression for the corresponding particle-particle susceptibility  $\chi^{\text{pp}}(\omega)$ .

To begin with, let us consider the non-interacting particle-hole and particle-particle bubbles involving one  $c$ -fermion and one  $d$ -electron propagator,

$$\Pi_0^{\text{ph}}(i\bar{\omega}) = \int_{\omega} G_0^c(i\omega) G_0^d(i\omega - i\bar{\omega}), \quad (33)$$

$$\Pi_0^{\text{pp}}(i\bar{\omega}) = \int_{\omega} G_0^c(i\omega) G_0^d(-i\omega + i\bar{\omega}). \quad (34)$$

Both bubbles exhibit logarithmic singularities. To extract these, we note that it is sufficient to retain only the low-energy part of the local  $c$ -fermion Green function, Eq. (16)

$$G_0^c(i\omega) \approx -i\pi\nu_0 \text{sgn}\omega \Theta(\xi_0 - |\omega|). \quad (35)$$

At vanishing temperature the integrations in Eqs. (33) and (34) can then be performed exactly and we obtain

$$\Pi_0^{\text{ph}}(i\bar{\omega}) = -\nu_0 \ln \left[ \frac{\xi_0}{i\bar{\omega} + \xi_d} \right], \quad (36)$$

$$\Pi_0^{\text{pp}}(i\bar{\omega}) = \nu_0 \ln \left[ \frac{\xi_0}{i\bar{\omega} - \xi_d} \right]. \quad (37)$$

Due to the divergence in both channels, a simple one-channel truncation is not sufficient to calculate the X-ray response. The parquet method is a systematic tool to resum the leading divergencies in both channels. There are different implementations of this method. Our FRG formulation is closely related to the particular implementation of the parquet renormalization group discussed by Maiti and Chubukov<sup>26</sup>, who represent all potentially important fluctuations via bosonic Hubbard-Stratonovich fields, calculate the renormalization of the associated three-legged vertices, and finally use these vertices to calculate the susceptibilities. In order to formulate this program within the framework of the FRG, we bosonize the interactions in the two relevant channels. To do this, it is sufficient to retain only those interaction processes which mediate the singular interactions in the particle-hole and particle-particle channels. To derive the corresponding channel decomposition, we note that in frequency space the interaction in Eq. (10) can be written in the following three equivalent ways,

$$\begin{aligned} S_{\text{int}} &\equiv U \int_0^\beta d\tau \bar{c}(\tau) c(\tau) \bar{d}(\tau) d(\tau) \\ &= U \int_{\omega'_1} \int_{\omega'_2} \int_{\omega_2} \int_{\omega_1} \delta_{\omega'_1 + \omega'_2, \omega_2 + \omega_1} \bar{c}_{\omega'_1} \bar{d}_{\omega'_2} d_{\omega_2} c_{\omega_1} \\ &= U \int_{\omega} \int_{\omega'} \int_{\bar{\omega}} (\bar{c}_{\omega + \bar{\omega}} c_{\omega}) (\bar{d}_{\omega'} d_{\omega' + \bar{\omega}}) \\ &\quad (\text{frequency transfer } \bar{\omega} = \omega'_1 - \omega_1) \end{aligned} \quad (38a)$$

$$\begin{aligned} &= -U \int_{\omega} \int_{\omega'} \int_{\bar{\omega}} (\bar{c}_{\omega + \bar{\omega}} d_{\omega}) (\bar{d}_{\omega'} c_{\omega' + \bar{\omega}}) \\ &\quad (\text{exchange frequency } \bar{\omega} = \omega_1 - \omega'_2) \end{aligned} \quad (38b)$$

$$\begin{aligned} &= U \int_{\omega} \int_{\omega'} \int_{\bar{\omega}} (\bar{c}_{\omega + \bar{\omega}} \bar{d}_{-\omega}) (d_{-\omega'} c_{\omega' + \bar{\omega}}) \\ &\quad (\text{total frequency } \bar{\omega} = \omega_1 + \omega_2). \end{aligned} \quad (38c)$$

The last three lines can be generated from each other by re-labelling the frequencies. However, if we impose an ultraviolet cutoff  $|\bar{\omega}| < \Lambda_0 \ll \xi_0$  on the bosonic frequency, each of the above expressions describes a different low-energy scattering process: forward scattering (38a), exchange scattering (38b), and Cooper scattering (38c). The forward scattering channel can be ignored for our purpose because the corresponding susceptibility does not exhibit any singularity. We therefore retain only the exchange and the Cooper channels for small values of the corresponding energies, which amounts to retaining only the following low-energy terms in the effective interaction,

$$S_{\text{int}} \approx \int_{\bar{\omega}} \Theta(\Lambda_0 - |\bar{\omega}|) [-U_x \bar{A}_{\bar{\omega}} A_{\bar{\omega}} + U_p \bar{B}_{\bar{\omega}} B_{\bar{\omega}}], \quad (39)$$

where we have defined the two composite fields,

$$A_{\bar{\omega}} = \int_{\omega} \bar{d}_{\omega} c_{\omega + \bar{\omega}}, \quad (40a)$$

$$B_{\bar{\omega}} = \int_{\omega} d_{-\omega} c_{\omega + \bar{\omega}}. \quad (40b)$$

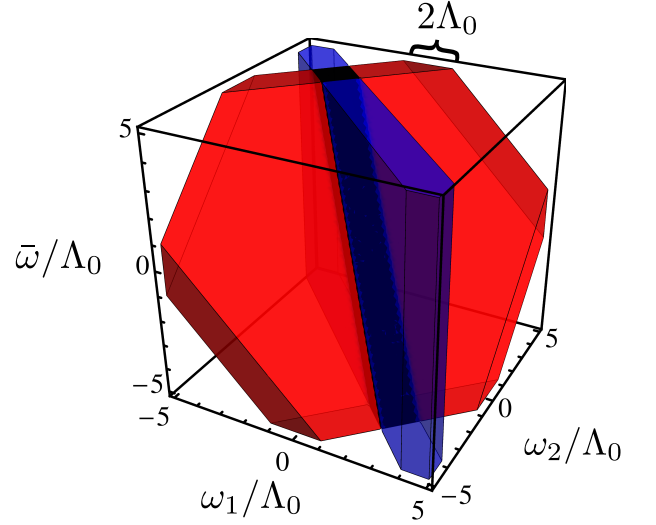


FIG. 2. (Color online) Low-energy sectors in the three dimensional frequency space spanned by the two frequencies  $\omega_1$  and  $\omega_2$  of the incoming fermions and the frequency transfer  $\bar{\omega} = \omega'_1 - \omega_1$ . The blue sector perpendicular to the  $\omega_1 - \omega_2$  plane represents the regime  $|\omega_1 + \omega_2| < \Lambda_0$  responsible for the dominant renormalization of effective interaction in the Cooper channel, while the red sector defined by  $|\omega'_2 - \omega_1| < \Lambda_0$  is responsible for the dominant renormalization in the exchange scattering channel.

Although  $U_x = U_p = U$  we have introduced two different coupling constants to distinguish the bare interaction  $U_x$  in the exchange channel from the interaction  $U_p$  in the Cooper (or pairing) channel. Also note that a certain small sector of the three-dimensional frequency space formed by  $\omega_1, \omega_2$  and  $\bar{\omega} = \omega'_1 - \omega_1$  is counted twice in Eq. (39), because the sectors defined by  $|\omega'_2 - \omega_1| < \Lambda_0$  and  $|\omega_2 + \omega_1| < \Lambda_0$  have a finite overlap proportional to  $\Lambda_0^2$ , corresponding to the intersection of the two frequency slices shown in Fig. 2. However, the phase space of these processes is proportional to  $\Lambda_0^2$  and can be neglected if a physical quantity is dominated by generic frequencies. Finally, we decouple the interaction using two complex bosonic Hubbard-Stratonovich fields  $\chi$  and  $\psi$  and obtain for the cutoff-dependent effective low-energy action

$$\begin{aligned} S_{\text{eff}} &= - \int_{\bar{\omega}} [(G_0^c(i\omega))^{-1} \bar{c}_{\omega} c_{\omega} + (G_0^d(i\omega))^{-1} \bar{d}_{\omega} d_{\omega}] \\ &\quad + \int_{\bar{\omega}}^{\Lambda_0} [U_x^{-1} \bar{\chi}_{\bar{\omega}} \chi_{\bar{\omega}} + \gamma_{x0} (\bar{A}_{\bar{\omega}} \chi_{\bar{\omega}} + A_{\bar{\omega}} \bar{\chi}_{\bar{\omega}})] \\ &\quad + \int_{\bar{\omega}}^{\Lambda_0} [U_p^{-1} \bar{\psi}_{\bar{\omega}} \psi_{\bar{\omega}} + i\gamma_{p0} (\bar{B}_{\bar{\omega}} \psi_{\bar{\omega}} + B_{\bar{\omega}} \bar{\psi}_{\bar{\omega}})], \end{aligned} \quad (41)$$

where the bare values of the Yukawa vertices are  $\gamma_{x0} = \gamma_{p0} = 1$ , and we have introduced the notation  $\int_{\bar{\omega}}^{\Lambda_0} = \int_{\bar{\omega}} \Theta(\Lambda_0 - |\bar{\omega}|)$ .

It is now straightforward to write down formally exact FRG flow equations of the above theory for the irreducible vertices<sup>35,39</sup>. For our purpose, it is sufficient to work with a truncation of these flow equations where

only irreducible vertices with two and three external legs are retained. Using the same cutoff scheme as in Sec. III (i.e., we introduce a cutoff only into the  $d$ -electron propagator), the cutoff-dependent propagators  $F_\Lambda^X(i\bar{\omega})$  and  $F_\Lambda^\psi(i\bar{\omega})$  of our bosonic fields are of the form

$$F_\Lambda^X(i\bar{\omega}) = \frac{1}{U_x^{-1} + \Pi_\Lambda^{\text{ph}}(i\bar{\omega})}, \quad (42)$$

$$F_\Lambda^\psi(i\bar{\omega}) = \frac{1}{U_p^{-1} + \Pi_\Lambda^{\text{pp}}(i\bar{\omega})}, \quad (43)$$

where  $\Pi_\Lambda^{\text{ph}}(i\bar{\omega})$  and  $\Pi_\Lambda^{\text{pp}}(i\bar{\omega})$  are the cutoff-dependent irreducible particle-hole and particle-particle bubbles. These can be identified with the self-energies of our Hubbard-Stratonovich fields. From the formally exact FRG flow equation for the generating functional of the one-line irreducible vertices<sup>35,36,39,40</sup> it is now straightforward to write down formally exact FRG flow equations for the vertices of our coupled fermion-boson theory. Neglecting the mixed four-point vertices with two bosonic and two

fermionic external legs (these vertices vanish at the initial scale and are expected to remain small at least in the weak coupling regime) the bosonic self-energies satisfy the truncated FRG flow equations

$$\begin{aligned} \partial_\Lambda \Pi_\Lambda^{\text{ph}}(i\bar{\omega}) &= \int_\omega G_\Lambda^c(i\omega) \dot{G}_\Lambda^d(i\omega - i\bar{\omega}) \\ &\times \Gamma_\Lambda^{\bar{c}d\chi}(\omega, \omega - \bar{\omega}, \bar{\omega}) \Gamma_\Lambda^{\bar{d}c\bar{\chi}}(\omega - \bar{\omega}, \omega, \bar{\omega}), \end{aligned} \quad (44a)$$

$$\begin{aligned} \partial_\Lambda \Pi_\Lambda^{\text{pp}}(i\bar{\omega}) &= - \int_\omega G_\Lambda^c(i\omega) \dot{G}_\Lambda^d(-i\omega + i\bar{\omega}) \\ &\times \Gamma_\Lambda^{\bar{c}d\psi}(\omega, \bar{\omega} - \omega, \bar{\omega}) \Gamma_\Lambda^{\bar{d}c\bar{\psi}}(\bar{\omega} - \omega, \omega, \bar{\omega}). \end{aligned} \quad (44b)$$

A graphical representation of these flow equations is shown in Fig. 3 (a) and (b). The four different Yukawa vertices, appearing in Eqs. (44a) and (44b), satisfy the flow equations shown graphically in Fig. 3 (c) and (d). The flow equations for the Yukawa vertices are explicitly given by

$$\begin{aligned} \partial_\Lambda \Gamma_\Lambda^{\bar{c}d\chi}(\omega'_1, \omega_1, \bar{\omega}_2) &= \int_{\bar{\omega}} F_\Lambda^\psi(i\bar{\omega}) \dot{G}_\Lambda^d(-i\omega'_1 + i\bar{\omega}) G_\Lambda^c(-i\omega_1 + i\bar{\omega}) \\ &\times \Gamma_\Lambda^{\bar{c}d\bar{\psi}}(\omega'_1, -\omega'_1 + \bar{\omega}, \bar{\omega}) \Gamma_\Lambda^{\bar{d}c\bar{\psi}}(\omega_1, -\omega_1 + \bar{\omega}, \bar{\omega}) \Gamma_\Lambda^{\bar{d}c\chi}(-\omega_1 + \bar{\omega}, -\omega'_1 + \bar{\omega}, \bar{\omega}_2), \end{aligned} \quad (45a)$$

$$\begin{aligned} \partial_\Lambda \Gamma_\Lambda^{\bar{d}c\bar{\chi}}(\omega'_1, \omega_1, \bar{\omega}_2) &= \int_{\bar{\omega}} F_\Lambda^\psi(i\bar{\omega}) G_\Lambda^c(-i\omega'_1 + i\bar{\omega}) \dot{G}_\Lambda^d(-i\omega_1 + i\bar{\omega}) \\ &\times \Gamma_\Lambda^{\bar{c}d\bar{\psi}}(-\omega'_1 + \bar{\omega}, \omega'_1, \bar{\omega}) \Gamma_\Lambda^{\bar{d}c\bar{\psi}}(-\omega_1 + \bar{\omega}, \omega_1, \bar{\omega}) \Gamma_\Lambda^{\bar{d}c\bar{\chi}}(-\omega_1 + \bar{\omega}, -\omega'_1 + \bar{\omega}, \bar{\omega}_2), \end{aligned} \quad (45b)$$

$$\begin{aligned} \partial_\Lambda \Gamma_\Lambda^{\bar{c}d\psi}(\omega'_1, \omega'_2, \bar{\omega}_1) &= - \int_{\bar{\omega}} F_\Lambda^X(i\bar{\omega}) \dot{G}_\Lambda^d(i\omega'_1 - i\bar{\omega}) G_\Lambda^c(i\omega'_2 + i\bar{\omega}) \\ &\times \Gamma_\Lambda^{\bar{d}c\chi}(\omega'_1, \omega'_1 - \bar{\omega}, \bar{\omega}) \Gamma_\Lambda^{\bar{d}c\bar{\chi}}(\omega'_2, \omega'_2 + \bar{\omega}, \bar{\omega}) \Gamma_\Lambda^{\bar{c}d\psi}(\omega'_2 + \bar{\omega}, \omega'_1 - \bar{\omega}, \bar{\omega}_1), \end{aligned} \quad (45c)$$

$$\begin{aligned} \partial_\Lambda \Gamma_\Lambda^{\bar{d}c\bar{\psi}}(\omega_1, \omega_2, \bar{\omega}_1) &= - \int_{\bar{\omega}} F_\Lambda^X(i\bar{\omega}) G_\Lambda^c(i\omega_1 + i\bar{\omega}) \dot{G}_\Lambda^d(i\omega_2 - i\bar{\omega}) \\ &\times \Gamma_\Lambda^{\bar{c}d\chi}(\omega_1 + \bar{\omega}, \omega_1, \bar{\omega}) \Gamma_\Lambda^{\bar{d}c\bar{\chi}}(\omega_2 - \bar{\omega}, \omega_2, \bar{\omega}) \Gamma_\Lambda^{\bar{d}c\bar{\psi}}(\omega_1 + \bar{\omega}, \omega_2 - \bar{\omega}, \bar{\omega}_1). \end{aligned} \quad (45d)$$

Note that in our cutoff scheme where only the  $d$ -electron propagator is regularized the FRG equation for the fermionic self-energy is still given by Eq. (19). To recover the leading order parquet results for the X-ray response it is sufficient to set all frequency dependencies of the Yukawa vertices equal to zero. In this limit

$$\Gamma^{\bar{c}d\chi}(0, 0, 0) = \Gamma^{\bar{d}c\bar{\chi}}(0, 0, 0) \equiv \gamma_x, \quad (46a)$$

$$\Gamma^{\bar{c}d\psi}(0, 0, 0) = \Gamma^{\bar{d}c\bar{\psi}}(0, 0, 0) \equiv i\gamma_p. \quad (46b)$$

Ignoring all self-energy corrections (fermionic and bosonic) to the right-hand sides of the FRG flow equa-

tions we find

$$\partial_l \Pi^{\text{ph}}(0) = -\nu_0 \gamma_x^2, \quad (47a)$$

$$\partial_l \Pi^{\text{pp}}(0) = \nu_0 \gamma_p^2, \quad (47b)$$

$$\partial_l \gamma_x = \nu_0 U_p \gamma_p^2 \gamma_x, \quad (47c)$$

$$\partial_l \gamma_p = -\nu_0 U_x \gamma_x^2 \gamma_p, \quad (47d)$$

where  $\partial_l = -\Lambda \partial_\Lambda$  is the logarithmic scale derivative. Setting  $u_x = \nu_0 U_x$  and  $u_p = \nu_0 U_p$  and defining  $g_x = \gamma_x^2$  and  $g_p = \gamma_p^2$ , the last two equations can be written as

$$\partial_l g_x = 2u_p g_x g_p, \quad (48a)$$

$$\partial_l g_p = -2u_x g_x g_p, \quad (48b)$$

which should be solved with the initial conditions  $g_x(l = 0) = g_p(l = 0) = 1$ . Using the conservation law  $\partial_l [u_x g_x(l) + u_p g_p(l)] = 0$ , we can easily solve the two

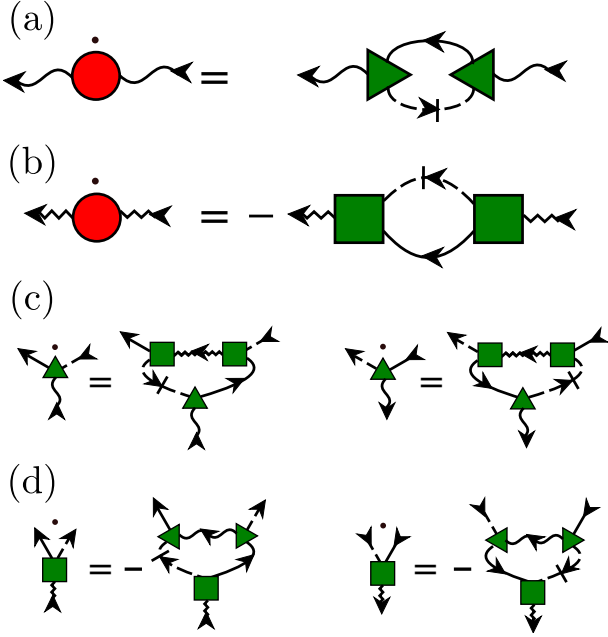


FIG. 3. (Color online) Diagram (a) represents the FRG flow equation (44a) for the self-energy of the Hubbard-Stratonovich field  $\chi$  (wavy line) representing particle-hole fluctuations in the exchange channel. Note that in our cut-off scheme only the  $d$ -electron propagator is regularized. The shaded triangles and squares denote the renormalized Yukawa vertices, while the other symbols are explained in the caption of Fig. 1. Diagram (b) represents the FRG flow equation (44b) for the self-energy of the pairing field  $\psi$  (zig-zag line). In (c) and (d) we show the FRG flow equations (45a–45d) for the Yukawa vertices associated with the particle-hole and the particle-particle channel.

coupled differential equations (48a) and (48b) exactly,

$$g_x(l) = \frac{(u_x + u_p)e^{(u_x + u_p)l}}{u_x e^{(u_x + u_p)l} + u_p e^{-(u_x + u_p)l}}, \quad (49a)$$

$$g_p(l) = \frac{(u_x + u_p)e^{-(u_x + u_p)l}}{u_x e^{(u_x + u_p)l} + u_p e^{-(u_x + u_p)l}}. \quad (49b)$$

For simplicity, let us now set  $u_x = u_p \equiv u$ , so that the flow of the vertices reduces to

$$g_x(l) = \frac{e^{2ul}}{\cosh(2ul)}, \quad (50a)$$

$$g_p(l) = \frac{e^{-2ul}}{\cosh(2ul)}. \quad (50b)$$

A graph of these functions as a function of the logarithmic flow parameter  $l$  is shown in Fig. 4. Substituting our explicit expressions (50a, 50b) for the scale dependent vertices into Eqs. (47a) and (47b) we obtain

$$\partial_l \Pi^{\text{pp}}(0) = -\nu_0 \frac{e^{2ul}}{\cosh(2ul)}, \quad (51a)$$

$$\partial_l \Pi^{\text{ph}}(0) = \nu_0 \frac{e^{-2ul}}{\cosh(2ul)}. \quad (51b)$$

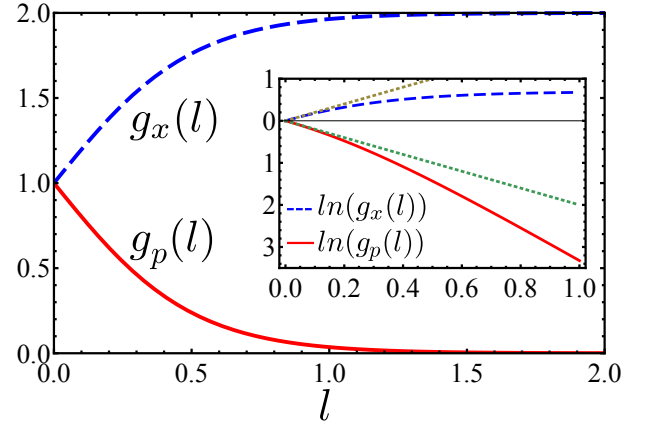


FIG. 4. (Color online) Graph of the scale dependent vertex factors  $g_x(l) = \gamma_x^2(l)$  and  $g_p(l) = \gamma_p^2(l)$  given in (50a, 50b) as a function of the logarithmic scale factor  $l = \ln(\Lambda_0/\Lambda)$ . In the inset we show  $\ln g_x(l)$  and  $\ln g_p(l)$  which in the leading order approximation are replaced by the straight dashed lines.

The crucial point is now that our leading order low-energy expansion of the frequency dependent susceptibilities is only valid as long as  $|\omega| \lesssim \Lambda = \Lambda_0 e^{-l}$ . To obtain the leading frequency dependence of the susceptibilities, we should therefore integrate the above equations up to the logarithmic scale factor  $l_* = \ln(\Lambda_0/|\bar{\omega}|)$ . Moreover, assuming  $ul_* \lesssim 1$  we may replace the factor  $\cosh(2ul)$  in the denominator of Eqs. (51a) and (51b) by unity. This corresponds to the leading order parquet approximation discussed by Roulet, Gavoret, and Nozières<sup>5</sup>. In this approximation we obtain

$$\begin{aligned} \Pi_{l_*}^{\text{ph}}(0) &= -\nu_0 \int_0^{l_*} dl e^{2ul} = \frac{\nu_0}{2u} [1 - e^{2ul_*}] \\ &= \frac{\nu_0}{2u} \left[ 1 - \left( \frac{\Lambda_0}{|\bar{\omega}|} \right)^{2u} \right], \end{aligned} \quad (52)$$

$$\begin{aligned} \Pi_{l_*}^{\text{pp}}(0) &= \nu_0 \int_0^{l_*} dl e^{-2ul} = \frac{\nu_0}{2u} [1 - e^{-2ul_*}] \\ &= \frac{\nu_0}{2u} \left[ 1 - \left( \frac{|\bar{\omega}|}{\Lambda_0} \right)^{2u} \right]. \end{aligned} \quad (53)$$

Finally, we perform an analytic continuation to the real frequency axis and obtain for the singular part of the retarded particle-hole response function for positive frequencies,

$$\begin{aligned} \chi^{\text{ph}}(\omega) &= -\text{Re} \left[ \Pi_{l_*}^{\text{ph}}(0) \Big|_{i\bar{\omega} \rightarrow \omega + i0^+} \right] \\ &= \frac{\nu_0}{2u} \left[ \left( \frac{\Lambda_0}{\omega} \right)^{2u} - 1 \right]. \end{aligned} \quad (54)$$

Identifying  $\Lambda_0 = \xi_0$  and keeping in mind that we have set the threshold frequencies equal to zero, we reproduce the result (7) which has first been obtained by Mahan<sup>29</sup> and

later by Roulet, Gavoret, and Nozières<sup>5</sup> using conventional parquet methods. Within our FRG formalism, it is straightforward to calculate also the leading singularity of the particle-particle response function; we obtain

$$\begin{aligned}\chi^{\text{PP}}(\omega) &= \text{Re} \left[ \Pi_{l_*}^{\text{PP}}(0) \Big|_{i\bar{\omega} \rightarrow \omega + i0^+} \right] \\ &= \frac{\nu_0}{2u} \left[ 1 - \left( \frac{\omega}{\Lambda_0} \right)^{2u} \right].\end{aligned}\quad (55)$$

Note that the leading frequency dependence vanishes in a non-analytic way for  $\omega \rightarrow 0$ . We have not been able to find Eq. (55) anywhere in the published literature on the X-ray problem.

## V. CONCLUSIONS

In this work we have shown how to obtain the threshold behavior of the deep hole Green function as well as the particle-hole and particle-particle response functions in the X-ray problem using the functional renormalization group. While our FRG results are equivalent to the leading order parquet approximation<sup>5</sup>, the calculational effort within the framework of the FRG is much lower than in the traditional parquet approach. Technically, a novel feature of our approach is the use of multi-component Hubbard-Stratonovich fields to take into account the singularities in competing scattering channels. Note that recently several authors have proposed parametrizations of the momentum- and frequency-dependent four-point ver-

tex in a purely fermionic formulation of the FRG which take the special combinations of momenta and frequencies associated with bosonic collective modes<sup>41–47</sup> into account. The method based on multi-channel Hubbard-Stratonovich fields proposed here is an alternative to these purely fermionic implementations of the FRG. In fact, our approach is similar in spirit to the version of the parquet renormalization group discussed in Ref. [26], which is also based on the explicit introduction of fermion-boson vertices via Hubbard-Stratonovich transformations. We believe that our method will also be useful in the context of other problems where many-body interactions lead to competing instabilities in more than one channel. For example, in two-dimensional Fermi systems the low-energy scattering processes of interacting fermions can be classified into forward-, exchange- and Cooper scattering<sup>48–50</sup>, which can be bosonized by introducing three different Hubbard-Stratonovich fields. It is straightforward to write down the coupled FRG flow equations for the irreducible two-point and three-point vertices for this problem. The corresponding FRG flow will be discussed elsewhere. Finally, let us point out that with the FRG it is straightforward to go beyond the leading order parquet approximation by taking self-energy corrections to the propagators and higher order irreducible vertices into account.

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